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A SEPARATION OF CARDINALS

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Abstract

In this paper, we find a set theoretic condition of separating regular from singular cardinals.

1. Introduction

Ordinals and cardinals are two of the most fundamental notions in set theory, and of the most useful tools in topology. We have seen separation of cardinals via topological properties, i.e., certain topological properties are valid only if the cardinals used are regular ([1, 3]), or only if the cardinals used are singular ([2, 4]).

In this paper, we present a separation of cardinals independent of any other area of mathematics but set theory itself.

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2. Main Results

In this section, at first, we present two lemmas, which are necessary for our main results.

Lemma 2.1. *Let X be a set and U_* be a cover of subsets of X with $|U_*| = k$, such that there is no subcover of U_* with cardinality less than k. Then, there is a subset S of X with $|S| = cf(k)$ that cannot be covered by less than $cf(k)$ elements of U_*.*

Proof. Let $U_* = \{U_\gamma | \gamma \in k\}$ and $\lambda = cf(k)$, choose ordinals k_β, $\beta < \lambda$ with $\sup\{k_\beta\} = k$. For every $\beta < \lambda$, let $V_\beta = \bigcup\{U_\gamma | \gamma < k_\beta\}$ and $V_* = \{V_\beta | \beta < \lambda\}$, then V_* is a cover of X with $|V_*| = \lambda = cf(k)$, with no subcover of cardinality less than λ. For every $\beta < \lambda$, choose a point $x_\beta \in V_{\beta+1} \setminus V_\beta$, then the set of points $\{x_\beta | \beta < \lambda\}$ has cardinality λ, it cannot be covered by less than λ elements of V_*, and therefore, it cannot be covered by less than λ elements of U_*.

The proof of the lemma is complete.

The following lemma is an immediate consequence of the above:

Lemma 2.2. *Let X be a set and U_* be a cover of subsets of X with $|U_*| = k$, where k is a regular cardinal, and there is no subcover of U_* with cardinality less than k. Then, there is a subset S of X with $|S| = k$ that cannot be covered by less than k elements of U_*.*

Remark 2.1. The above lemma is not valid for singular cardinals, as we shall see in the following example:

Example 2.1. Let k be a singular cardinal with $cf(k) = \lambda$, choose regular cardinals k_β, $\beta < \lambda$ with $\sup\{k_\beta\} = k$. Consider the products

$$X_\delta = \prod_{\delta<\beta<\lambda} k_\beta,$$

and the disjoint union

$$X = \coprod_{\delta<\lambda} X_\delta.$$

Let $[0, a)$ be an initial segment of a cardinal k_γ, consider the sets of the form

$$\coprod_{\delta<\gamma^+} \left(\prod_{\delta<\beta<\lambda} k_\beta \times [0, a) \times \prod_{\gamma<\beta'<\lambda} k_{\beta'} \right) \coprod_{\gamma^+<\delta'<\lambda} \prod_{\delta'\leq\beta'<\lambda} k_{\beta'}.$$

These sets form a cover of X, the cardinality of this cover is clearly k, and has no subcover of cardinality less than k, since each component X_δ cannot be covered by less than k_δ elements of the cover. Let S be a subset of X with $|S| = k$.

Let T be the subset of S contained in some $\prod_{\delta<\beta<\lambda} k_\beta$, assume the "worst" case $|T| = k$, let us express T in the form $T = \cup\{T_\beta | \beta < \lambda\}$, if $\beta < \gamma$, then $T_\beta \subset T_\gamma$ and $|T_\beta| = k_\beta$. Then for any $T_{\beta''}$, there exists a cardinal $k_{\gamma'}$ such that $T_{\beta''}$ is contained in the set $\prod_{\delta<\beta<\lambda} k_\beta \times [0, k_{\gamma'}) \times \prod_{\gamma<\beta'<\lambda} k_{\beta'}$, since all k_γ's are all regular, so T is contained in λ elements of the cover restricted to $\prod_{\delta<\beta<\lambda} k_\beta$, and since X has λ "components", S is contained in λ elements of the cover.

Remark 2.2. So, we see that eventhough this space cannot be covered by less than k elements of the cover, any subset of this set, of cardinality k is covered by λ elements of the cover. The reason, we made such a complicated construction of the cover, is because someone might think that, perhaps given a cover U_* with $|U_*| = k$, it could be reduced to a cover that has a property similar to the one described in Lemma 2.2 for regular cardinals. The cover described in the example cannot be reduced to such a subcover.

Now, based on Lemma 2.2 and Example 2.1, we can state the theorem that separates cardinals.

Theorem 2.1. *Let k be a cardinal, the followings are equivalent*:

(i) *k is regular*.

(ii) *For every set X with* $|X| > k$, *and for every cover* U_* *with* $|U_*| = k$, *which has no subcover of cardinality less than k, there exists a subset S of X with* $|S| = k$, *that cannot be covered by less than k elements of* U_*.

We finish with the following observation:

Corollary 2.1. *Let X be a set,* U_* *be a cover of X with* $|U_*| = k$ *singular, with no subcover of cardinality less than k. Assume that every subset S of X with* $|S| = k$ *is covered by less than k elements of* U_*. *Then, there is a cardinal* $\mu \geq cf(k), \mu < k$ *such that every subset of X of cardinality k is covered by at most* μ *elements of* U_*.

Proof. Let $\lambda = cf(k)$, choose cardinals $k_\beta, \beta < \lambda$ with $\sup\{k_\beta\} = k$. Assume that for every k_β, there exists a subset S_β of X with $|S_\beta| = k$ that cannot be covered by k_β elements of U_*, then $\bigcup\{S_\beta | \beta < \lambda\}$ is covered by k elements of U_*, that contradicts the hypothesis. Therefore, there exists a cardinal μ such that every subset of X of cardinality k is covered by at most μ elements of U_*. The fact $\mu \geq cf(k)$ follows from Lemma 2.1. The proof of the corollary is complete.

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